

Haag's Theorem in Noncommutative Quantum Field Theory

K. V. Antipin^a, M.N. Mnatsakanova^b and Yu. S. Vernov^c

^a*Faculty of Physics, Moscow State University, Moscow, Russia*

^b*Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, Russia*

^c*Institute for Nuclear Research, RAS, Moscow, Russia*

Abstract

Haag's theorem was extended to noncommutative quantum field theory in a general case when time does not commute with spatial variables. It was proven that if S-matrix is equal to unity in one of two theories related by unitary transformation, then the corresponding one in another theory is equal to unity as well. In fact, this result is valid in any $SO(1, 1)$ invariant quantum field theory, of which an important example is noncommutative quantum field theory.

Pacs 11.10.Cd, 11.10.Nx.

1 Introduction

In the present paper we consider Haag's theorem - one of the most important results of axiomatic approach in quantum field theory [1, 2]. In the usual Hamiltonian formalism it is assumed that asymptotic fields are related with interacting fields by unitary transformation. Haag's theorem shows that in accordance with Lorentz invariance of the theory the interacting fields are also trivial which means that corresponding S-matrix is equal to unity. So the usual interaction representation can not exist in the Lorentz invariant theory. Let us recall that Haag's theorem considers two theories, in which quantum field operators at equal time as well as corresponding vacuum states Ψ_0^i are related by the unitary operator:

$$\begin{aligned}\varphi_f^2(t) &= V\varphi_f^1(t)V^{-1} \\ \Psi_0^2 &= CV\Psi_0^1, \quad C \in \mathbb{C}, \quad |C| = 1\end{aligned}\tag{1}$$

In accordance with the Haag's theorem four first Wightman functions coincide in two theories in usual Lorentz invariant case [1, 2]. Let us recall that by definition Wightman functions are:

$$W = W(x_1, \dots, x_n) = (\Psi_0, \varphi(x_1) \dots \varphi(x_n) \Psi_0)\tag{2}$$

The main consequence of the Haag's theorem is that from triviality of one of the fields in question it follows that other one is trivial too, which implies that corresponding S-matrix is equal to unity. It is known that in axiomatic quantum field theory (QFT) there is no field operator defined in a point. Only the smoothed operators written symbolically as

$$\varphi_f = \int \varphi(t, x) f(x) d^3x dt,$$

where $f(x)$ are test functions, can be rigorously defined.

In the formulation of Haag's theorem it is assumed that the formal operators $\varphi(t, x)$ can be smeared only on the spatial variables.

In $SO(1, k)$ invariant theory, where $k \in \mathbb{N}$, it was proved that in two theories related by a unitary transformation the first $k + 1$ Wightman functions coincide [3]. Thus in $SO(1, 1)$ invariant theory, the most important example of which is noncommutative quantum field theory, only the first two Wightman functions coincide.

2 Noncommutative Quantum Field Theory

Noncommutative quantum field theory (NC QFT) being one of the generalizations of standard QFT has been intensively developed during the past years (for reviews, see [4, 5]). The present development in this direction is connected with the construction of noncommutative geometry [6] and new physical arguments in favour of such a generalization of QFT [7]. Essential interest in NC QFT is also due the fact that in some cases it is a low-energy limit of string theory [8].

The simplest and at the same time most studied version of noncommutative field theory is based on the following Heisenberg-like commutation relations between coordinates:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (3)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

It is very important that NC QFT can be also formulated in commutative space, if we replace the usual product of quantum field operators (strictly speaking, of the corresponding test functions) by the \star - (Moyal-type) product [4, 5].

Let us remind that the \star -product is defined as

$$\varphi(x) \star \varphi(y) = \exp \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) \varphi(x) \varphi(y) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right)^n \varphi(x) \varphi(y) \quad (4)$$

Evidently the series in equation (4) has to be convergent.

It was shown [9] that this series is convergent if the corresponding test functions belong to one of the Gelfand-Shilov spaces S^β with $\beta < \frac{1}{2}$. The similar result was obtained also in [10]. Moreover, $\varphi(x) \star \varphi(y)$ belongs to the same space S^β as $\varphi(x)$ and $\varphi(y)$ [9].

Noncommutative theories defined by Heisenberg-like commutation relations (3) can be divided into two classes.

The first of them is the case of only space-space noncommutativity, that is $\theta^{0i} = 0$, time commutes with spatial coordinates.

It is known that this case is free from the problems with causality and unitarity [11, 12, 13, 14, 15] and in this case the main axiomatic results: CPT and spin-statistics theorems, Haag's theorem remain valid [16] - [19].

Let us remind that if time commutes with spatial coordinates, then there exists one spatial coordinate, say x_3 , which commutes with all others. Simple calculations show that this result is valid also in a space with arbitrary even number of dimensions. Thus in space-space NC QFT we have two commuting coordinates and two noncommuting coordinates. For simplicity we consider four-dimensional case.

In the second case all coordinates, including time, are noncommuting.

Let us proceed to the LCC in noncommutative space-space QFT.

First let us recall this condition in commutative case. In the operator form this condition is

$$[\varphi_{f_1}, \varphi_{f_2}] = 0, \quad \text{if } O_1 \sim O_2, \quad (5)$$

where $O_1 = \text{supp } f_1, O_2 = \text{supp } f_2$. The condition $O_1 \sim O_2$ means that $(x - y)^2 < 0 \forall x \in O_1$ and $y \in O_2$.

In the noncommutative case we have the similar condition with respect to commutative coordinates, thus now $O_1 \sim O_2$ means that

$$(x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \quad (6)$$

3 Haag's theorem in space-space NC QFT

Let us recall that in NC QFT Lorentz invariance is broken up to $SO(1, 1) \otimes SO(2)$ symmetry [16]. As was already mentioned, in two $SO(1, 1)$ invariant theories related by a unitary transformation, only two-point Wightman functions coincide.

In what follows we prove that if one of considered theories is trivial, that is the corresponding S-matrix is equal to unity, then another is trivial too. To prove this in $SO(1, 1)$ invariant theory it is sufficient that the spectral condition, which implies non existence of tachyons, is valid only in respect with commutative coordinates that is

$$P_i^0 \geq |P_i^3| \quad (7)$$

for arbitrary state.

Also it is sufficient that translation invariance is valid only in respect with the commutative coordinates.

Let us point out that in case the spectral condition (7) and $SO(1, 1)$ symmetry are fulfilled, Wightman functions are analytical functions in two-dimensional analog of extended tubes [19].

The equality of two-point Wightman functions in two theories leads to the following conclusion: if local commutativity condition in respect with commutative coordinates is fulfilled and the current in one of the theories is equal to zero, then another current is zero as well.

Indeed as $W_1(x^1, x^2) = W_2(x^1, x^2)$, then also

$$\left(\Psi_0^1, j_f^1 j_f^1 \Psi_0^1\right) = \left(\Psi_0^2, j_f^2 j_f^2 \Psi_0^2\right) \quad (8)$$

where

$$j_f^i = (\square + m^2) \varphi_f^i.$$

If, for example, $j_f^1 = 0$, then in the space with positive metric

$$j_f^2 \Psi_0^2 = 0 \quad (9)$$

From the latter formula, corresponding analytical properties of Wightman functions, and local commutativity condition (5) it follows [2] that

$$j_f^2 \equiv 0 \quad (10)$$

Our statement is proved.

4 Haag's Theorem in General Case

Now let us consider the general case, when all coordinates, including time, do not commute. Let us stress that NC QFT in general case is $SO(1, 1)$ invariant as well as in the case of space-space noncommutativity [16].

In fact our derivation is valid in any theory with $SO(1, 1)$ symmetry, if corresponding test functions belong to one of the Gelfand-Shilov space S^β with $\beta < \frac{1}{2}$.

Wightman functions in noncommutative theory take the following form with the Moyal-type product:

$$W_\star(x_1, \dots, x_n) = (\Psi_0, \phi(x_1) \star \dots \star \phi(x_n) \Psi_0) \quad (11)$$

Corresponding generalized function on Gelfand- Shilov space S^β acts as

$$(\Psi_0, \phi_{f_1} \star \dots \star \phi_{f_n} \Psi_0) = \int W(x_1, \dots, x_n) f_1(x_1) \star \dots \star f_n(x_n) dx_1 \dots dx_n, \quad (12)$$

where $W(x_1, \dots, x_n) = (\Psi_0, \phi(x_1) \dots \phi(x_n) \Psi_0)$.

According to [9] expression $f_1(x_1) \star \dots \star f_n(x_n)$ is well-defined for S^β with $\beta < \frac{1}{2}$ and belongs to the same space S^β as $f_i(x_i)$.

Let us recall that the series for the Moyal-type product has the following form:

$$f_\star(x, y) \equiv \exp\left(\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}\right) f(x) f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}\right)^n f(x) f(y). \quad (13)$$

Here the expression is written only for two points x and y instead of x_1, \dots, x_n for simplicity.

In accordance with eq. (12) Wightman functions are generalized functions corresponding to test functions, which belong to S^β .

Let us consider the set of functions satisfying the condition

$$\frac{d^n}{dx^n} f_k(x) = 0, \quad \text{if } n > k. \quad (14)$$

Evidently these functions belong to S^β . In accordance with latter condition the expansion (13) contains a finite number of derivatives. Thus corresponding Wightman functions W_k are tempered distributions.

Now we can take advantage of using W_k : at any finite number k we have a local theory, that is a theory with finite maximal speed of interaction spread. Thus we can assume that just as in the case of space-space NC QFT quantum field operators satisfy local commutativity condition in respect with variables satisfying $SO(1,1)$ symmetry, that gives us possibility to use the same derivation of Haag's theorem as in the case of space-space NC QFT. Let us point out that maximal speed of interaction spread depends on k and goes to infinity if $k \rightarrow \infty$, as NC QFT is a theory with infinite speed of interaction spread.

Thus we have proved Haag's theorem for the special choice of Wightman functions. Then we can take the limit $k \rightarrow \infty$. Let us take into account that functions $f_k(x_1, \dots, x_n)$, satisfying eq. (14), owing to convergence of the set (13), go to the limit $f(x_1) \star \dots \star f(x_n)$. So the sequence of functionals W_k converges weakly to some functional W . It is known that Gelfand-Shilov space S^β is complete with respect to weak convergence (see [20], p. 91). Haag's theorem is valid for Wightman functions, coinciding with W_k , therefore it is valid for their limit value W corresponding to $k = \infty$. So Haag's theorem has been extended to general case of time-space noncommutativity.

5 Conclusions

We see that Haag's theorem is valid in any theory with $SO(1,1)$ symmetry, if corresponding test functions belong to one of the Gelfand-Shilov spaces S^β with $\beta < \frac{1}{2}$. The important physical example of such a theory is NC QFT.

References

- [1] R. F. Streater, A. S. Wightman, *PCT, spin and statistics, and all that* (Benjamin, New York 1964).
- [2] N. N. Bogoliubov, A. A. Logunov, I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, Reading, Mass 1975).
- [3] K. V. Antipin, M. N. Mnatsakanova, Yu. S. Vernov, PoS - Proceedings of Science, SISSA, Trieste, Italy **080** (2010), [[math-ph/1102.1195](#)].

- [4] M. R. Douglas, N. A. Nekrasov, *Rev. Mod. Phys.* **73**, 977 (2001), [[hep-th/0106048](#)].
- [5] R. J. Szabo, *Phys. Rept.* **378**, 207 (2003), [[hep-th/0109162](#)].
- [6] A. Connes, *Noncommutative Geometry*, Academic Press, New York (1994).
- [7] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Phys. Lett. B* **331** 39 (1994); *Comm. Math. Phys.*, **172**, 187 (1995).
- [8] N. Seiberg, E. Witten, *JHEP* **9909**, 32 (1999), [[hep-th/9908142](#)].
- [9] M. Chaichian, M. Mnatsakanova, A. Tureanu, Yu. Vernov, *JHEP* **09**, 125 (2008), [[hep-th/0706.1712](#)].
- [10] M. A. Soloviev, *Theor. Math. Phys.* **153**, 1351 (2007), [math-ph/0708.0811](#).
- [11] J. Gomis and T. Mehen, *Nucl. Phys. B* **591** (2000) 265, [[hep-th/0005129](#)].
- [12] O. Aharony, J. Gomis, and T. Mehen, *JHEP* **0009** (2000) 023 [[hep-th/0006236](#)].
- [13] N. Seiberg, L. Susskind, and N. Toumbas, *JHEP* **0006** (2000) 044 [[hep-th/0005015](#)].
- [14] L. ' Alvarez-Gaum'e and J. L. F. Barbon, *Int. J. Mod. Phys. A* **16**, (2001)1123 [[hep-th/0006209](#)].
- [15] L. ' Alvarez-Gaum'e, J. L. F. Barbon, and R. Zwicky, *JHEP* **0105** (2001) 057 [[hep-th/0103069](#)].
- [16] L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *Nucl. Phys. B* **668**, 293 (2003), [[hep-th/0305093v2](#)].
- [17] M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A. Tureanu and Yu. S. Vernov, *J. Math. Phys.* **52**, 032303 (2011), [hep-th/0402212](#).
- [18] Yu.S. Vernov, M.N. Mnatsakanova, *Theor. Math. Phys.* **142**, 337 (2005).
- [19] M. Chaichian, M. N. Mnatsakanova, A. Tureanu and Yu. S. Vernov, [hep-th/0612112](#).
- [20] Gel'fand and Shilov, *Generalized Functions, V.2*, (Academic Press Inc., New York 1968), Chapter IV; (FIZMATGIZ, Moscow, 1958) (in Russian).